Tutorial 7: Selected problems of Assignment 6

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Q1) (Supp. Ex. 3)

Let $f : [a,b] \to \mathbb{R}$ be a nonnegative continuous function.

Show that $\int_a^b f = 0 \iff f \equiv 0$

Pf) [$\Leftarrow$] Trivial.

[$\Rightarrow$] Proof by contrapositive : Suppose $f \not\equiv 0$,

want to show $\int_a^b f \neq 0$.

Since $f \not\equiv 0$, there exists $x_0 \in [a,b]$ such that $f(x_0) > 0$

By continuity of f on $[a,b]$, there exists $\delta > 0$ such that

$$\forall x \in [x_0 - \delta, x_0 + \delta] \cap [a,b] =: I, \quad f(x) > \frac{f(x_0)}{2}$$

$$\therefore \int_a^b f \geq \int_I f \quad (\text{since } f \geq 0)$$

$$\geq \int_I \frac{f(x_0)}{2} \geq \frac{f(x_0)}{2} \cdot \delta > 0$$

$$\therefore \int_a^b f \neq 0.$$

$-\boxtimes$

Q2) (§7.2 Q18)

Let $f : [a,b] \to \mathbb{R}$ be a nonnegative continuous function.

Let $M_n := \left(\int_a^b f^n \right)^{\frac{1}{n}}$ and $M := \sup_{[a,b]} f$

Show that $\lim_n M_n = M$

Pf Case 1: $f \equiv 0$: then $M_n = 0 = M, \forall n \in \mathbb{N}$

Case 2: $f \not\equiv 0$: then by Max-Min Thm,

$\exists x_0 \in [a,b]$ s.t. $f(x_0) = M > 0$.

Given $0 < \varepsilon < f(x_0)$, by continuity of f, $\exists \delta > 0$ s.t.

$\forall x \in [x_0 - \delta, x_0 + \delta] \cap [a,b] =: I, \quad |f(x) - f(x_0)| < \varepsilon$

In particular, $f(x) > f(x_0) - \varepsilon > 0$

$\therefore \int_a^b f^n \geq \int_I f^n \geq \int_I (f(x_0) - \varepsilon)^n = (f(x_0) - \varepsilon)^n \cdot \delta$

$\therefore M_n = \left(\int_a^b f^n \right)^{\frac{1}{n}} \geq (f(x_0) - \varepsilon) \cdot \delta^{\frac{1}{n}}$

$\therefore \underline{\lim}_n M_n \geq (f(x_0) - \varepsilon) \lim_n \delta^{\frac{1}{n}} = f(x_0) - \varepsilon$

Letting $\varepsilon \to 0 \Rightarrow \underline{\lim}_n M_n \geq f(x_0) = M$

On the other hand: $\int_a^b f^n \leqslant \int_a^b M^n = M^n(b-a)$

$\therefore$ $M_n = \left(\int_a^b f^n\right)^{\frac{1}{n}} \leq M \cdot (b-a)^{\frac{1}{n}}$

$\therefore$ $\overline{\lim_n} M_n \leqslant M \cdot \overline{\lim_n} (b-a)^{\frac{1}{n}} = M$

Combining above: $M \leqslant \underline{\lim_n} M_n \leqslant \overline{\lim_n} M_n \leqslant M$

$\therefore$ $\lim_n M_n$ exists and $\lim_n M_n = M$ $\quad - \boxed{}$

Q3) (§7.2 Q19)

Let $f: [-a, a] \to \mathbb{R}$ be integrable.

if f is odd (i.e. $f(-x) = -f(x)$, $\forall x \in [0, a]$)

Show that $\int_{-a}^{a} f = 0$

Pf: Let $P_n = \{0, \frac{a}{n}, \frac{2}{n}a, \cdots, a\} =: \{x_0, x_1, \cdots, x_n\}$

be a partition on $[0, a]$ then $\|P_n\| \to 0$.

Choose $z_j = x_j \in [x_j, x_{j+1}]$ Therefore, by Theorem 2.6,

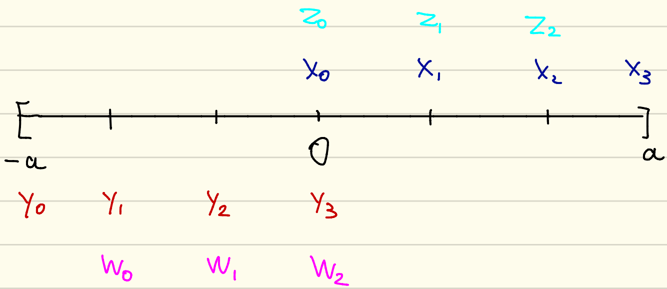
$$\int_0^a f = \lim_n S(f, \dot{P}_n) = \lim_n \sum_{j=0}^{n-1} f(z_j) \cdot \frac{a}{n}$$

On the other hand, let $Q_n = \{-a, -a + \frac{a}{n}, \cdots, -a + a = 0\}$

$= \{y_0, \cdots, y_n\}$ be a partition on $[-a, 0]$. Let $w_j = y_{j+1} \in [y_j, y_{j+1}]$

Then similar as above, $\int_{-a}^{0} f = \lim_n S(f; \dot{Q}_n) = \lim_n \sum_{k=0}^{n-1} f(w_k) \cdot \frac{a}{n}$

Auxillary diagram for notations : e.g. for $n=3$:



Now since f is odd, $\forall\ 0 \leq j \leq n-1$,

$$f(z_j) = f\left(\tfrac{j}{n}\cdot a\right) = -f\left(-\tfrac{j}{n}\cdot a\right) = -f\left(-a + \tfrac{(n-j)}{n}\cdot a\right) = -f(w_{n-1-j})$$

$$\therefore\ \forall n,\ \sum_{j=0}^{n-1} f(z_j)\cdot\frac{a}{n} = \sum_{j=0}^{n-1}\left(-f(w_{n-1-j})\right)\cdot\frac{a}{n} = -\sum_{k=0}^{n} f(w_k)\,\frac{a}{n}$$

Therefore, $\displaystyle\int_0^a f = \lim_n \sum_{j=0}^{n-1} f(z_j)\cdot\frac{a}{n} = -\lim_n \sum_{k=0}^{n-1} f(w_j)\cdot\frac{a}{n} = -\int_{-a}^0 f$

$$\therefore\ \int_{-a}^{a} f = \int_{-a}^{0} f + \int_0^a f = 0$$

Q4) (Supp. Ex. 8) Let $f:[a,b]\to\mathbb{R}$ be continuous

and $g:[a,b]\to\mathbb{R}$ be positive continuous

then there exists $c\in[a,b]$ s.t.

$$\int_a^b fg = f(c)\cdot\int_a^b g$$

Pf) Let $F(x)=\int_a^x fg$; $G(x)=\int_a^x g$

Then as f, g are continuous on $[a,b]$,

by Second Fundamental Theorem of Calculus (Thm 2.14)

F, G are continuous on $[a,b]$, differentiable on (a,b)

and $\forall x\in(a,b)$ $F'(x)=(fg)(x)$; $G'(x)=g(x)>0$

$\therefore$ By Cauchy Mean Value Theorem (Ch. 6 Thm. 6.3.2)

there exists $c\in(a,b)$ such that $\dfrac{F(b)-F(a)}{G(b)-G(a)}=\dfrac{F'(c)}{G'(c)}$

Note that LHS $=\dfrac{\int_a^b fg-0}{\int_a^b g-0}$ and RHS $=\dfrac{(fg)(c)}{g(c)}=f(c)$

$$\therefore \int_a^b fg = f(c)\int_a^b g$$